POINTWISE INNER AUTOMORPHISMS OF VON NEUMANN ALGEBRAS

by

Uffe Haagerup(*) and Erling Størmer(**)

with an appendix by

Colin Sutherland(***)

(*) Institut for matematik og datalogi,
Odense Universitet, DK-5230 Odense M, Denmark.

(**) Matematisk institutt,
Universitetet i Oslo, Blindern, 0316 Oslo 3, Norway.

(***) Department of Mathematics,
University of New South Wales, Kensington, NSW 2033 Australia.

Abstract

An automorphism α of a von Neumann algebra M is called pointwise inner if for all $\phi \in M_*^+$ there is a unitary $u \in M$ such that $\phi\circ\alpha = u\phi u^*$. We analyse such automorphisms; in particular we show that if M is a factor of type III_λ, $0\leq\lambda<1$, with separable predual, then an automorphism α is pointwise inner if and only if there are an inner automorphism γ and an extended modular automorphism $\bar{\sigma}_c^\omega$ in the sense of Connes and Takesaki, such that $\alpha = \gamma\circ\bar{\sigma}_c^\omega$.

Pointwise inner automorphisms of von Neumann algebras

by

Uffe Haagerup and Erling Størmer

1. Introduction

In our paper [7] we introduced two classes of automorphisms of von Neumann algebras. If $\alpha \in \mathrm{Aut}(M)$ for a von Neumann algebra M, then α was called pointwise inner if α preserves unitary equivalence classes of normal states, i.e. if for all ϕ in the positive part M_*^+ of the predual of M, there is $u = u(\phi)$ in the unitary group $U(M)$ of M such that

$$\phi \circ \alpha = u\phi u^*.$$

α was called approximately pointwise inner if α preserves norm closures of unitary equivalence classes in M_*^+, i.e. if for all $\phi \in M_*^+$ and $\varepsilon > 0$ there is $u = u(\phi,\varepsilon) \in U(M)$ such that

$$\| \phi \circ \alpha - u\phi u^* \| < \varepsilon.$$

The approximately pointwise inner automorphisms were studied to some extent in [7], while we only proved two results for the pointwise inner ones. One was that if M is semifinite with separable predual, then each pointwise inner automorphism of M is inner. This followed, as pointed out to us by V. Jones, from a result of Popa [9] on maximal abelian algebras. The other result was that each modular automorphism is pointwise inner.

We shall in the present paper give a rather complete description of the pointwise inner automorphisms for factors M of type III_λ, $0<\lambda<1$, with separable preduals. Our techniques rely heavily on the existence of faithful normal strictly semifinite lacunary weights of infinite multiplicity on such factors, hence we are unable to do anything in the III_1-case. Let ϕ be such a weight. Then we first show that an automorphism α of M is pointwise inner if and only if there is $v \in U(M)$ such that $\alpha o \mathrm{Ad}v$ is ϕ-invariant, and $\alpha o \mathrm{Ad}v|_{M_\phi} = \iota$ - the identity map, where M_ϕ is the centralizer of ϕ. We then restrict attention to ϕ-invariant α's for which $\alpha|_{M_\phi} = \iota$, and show that the group of such α's is isomorphic to the unitary group of the center C_ϕ of M_ϕ. Furthermore, α is inner if and only if its image is a coboundary with respect to the natural ergodic action induced on C_ϕ. This allows us to define an abelian cohomology group $H^1(\mathbb{Z},U(C_\phi))$ which is isomorphic to the quotient of the pointwise inner automorphisms by the inner ones. For example, this group is the circle group when M is of type III_λ, $0<\lambda<1$. If M is of type III_0, $H^1(\mathbb{Z},U(C_\phi))$ can also be described as the closure of the inner automorphisms of M implemented by unitaries in C_ϕ by the inner automorphisms in this group.

In [7] we showed that each $\alpha \in \mathrm{Aut}(M)$ has a natural extension to an automorphism $\tilde{\alpha}$ of the crossed product $M \times_{\sigma^\phi} \mathbb{R}$. It was shown that α is approximately pointwise inner if and only if $\tilde{\alpha}$ is the identity on the center of $M \times_{\sigma^\phi} \mathbb{R}$. In §5 we show that α is pointwise inner if and only if $\tilde{\alpha}$ is inner. Furthermore it follows from this that the pointwise inner automorphisms are exactly those of the form $\mathrm{Ad}u \circ \bar{\sigma}_c^\omega$ with $u \in U(M)$ and $\bar{\sigma}_c^\omega$ the extended modular automorphism of M defined by a dominant weight and a cocycle c in the flow of weights, as defined by Connes and Takesaki in [4]. This gives in particular a new proof of the isomorphism between $H^1(\mathbb{Z},U(C_\phi))$ and $H^1(F^M)$ proved in [4, Appendix] (see also [13]).

Finally we show in §6 that in the nonseparable case the situation is quite different, indeed for some factors of type II_1 there are pointwise inner automorphisms which are outer.

In an appendix due to C. Sutherland it will be shown that in the III_0-case the cohomology group $H^1(\mathbb{Z}, U(C_\phi))$, and hence $H^1(F^M)$, is nonsmooth in its natural Borel structure, hence is a very big space.

2 Weights and automorphisms

Recall that a faithful normal weight ϕ on a von Neumann algebra is called strictly semifinite if its restriction to its centralizer is a semifinite trace, cf. [2, Def. 3.1.5]. Throughout this section M will be a von Neumann algebra.

Lemma 2.1. Let ϕ be a strictly semifinite faithful normal weight on M and α a pointwise inner automorphism on M. Then there is $u \in U(M)$ such that $\phi o \alpha = u\phi u^*$.

Proof. Since ϕ is strictly semifinite there is an orthogonal family $(e_k)_{k\in I}$ of projections in M_ϕ with sum 1 such that $\phi(e_k) < \infty$, $k \in I$. Let $\phi_k = \phi(e_k \cdot)$. Then $\phi = \sum_{k\in I} \phi_k$ and $\mathrm{supp}(\phi_k) = e_k$. Since α is pointwise inner there is for each $k \in I$, $u_k \in U(M)$ such that $\phi_k o\alpha = u_k\phi_k u_k^*$. Then $\alpha^{-1}(e_k) = \mathrm{supp}(\phi_k o \alpha) = u_k e_k u_k^*$. Let $u = \sum_{k\in I} u_k e_k$. Then a straightforward computation shows that $u \in U(M)$, and since $e_k \in M_\phi$ for all $k \in I$, it follows that for $x \in M^+$ we have

$$\begin{aligned}\phi(u^*xu) &= \sum_{k\in I} \phi(e_k u_k^* x u_k)\\ &= \sum_{k\in I} u_k\phi_k u_k^*(x)\\ &= \sum_{k\in I} \phi_k o\alpha(x)\\ &= \phi \; o \; \alpha(x)\end{aligned}$$

QED.

If M is semifinite with separable predual every pointwise inner automorphism of M is inner by [7, Prop. 12.5]. In the nonseparable case we can only show the following.

Lemma 2.2. Suppose τ is a faithful normal semifinite trace on M. Then $\alpha \in \mathrm{Aut}(M)$ is pointwise inner if and only if for all $x \in M^+ \cap L^1(M,\tau)$ there is $u = u(x) \in U(M)$ such that $\alpha(x) = uxu^*$.

Proof. Suppose α is pointwise inner, and let $x \in M^+ \cap L^1(M,\tau)$. Let Let $\phi = \tau(x\cdot)$. Then $\phi \in M_*^+$, so there is $u \in U(M)$ such that $\phi o \alpha^{-1} = u\phi u^*$. By Lemma 2.1 τ is α-invariant, hence if $y \in M^+$, we have

$$\tau(\alpha(x)y) = \tau(x\alpha^{-1}(y)) = \phi(\alpha^{-1}(y)) = \phi(u^*yu) =$$
$$= \tau(xu^*yu) = \tau(uxu^*y).$$

It follows that $\alpha(x) = uxu^*$.

Conversely suppose such a u exists for each $x \in M^+ \cap L^1(M,\tau)$. Let $\phi \in M_*^+$, and let $h = \frac{d\phi}{d\tau} \in L^1(M,\tau)^+$. Then h is self-adjoint, and we put

$$h_n = \chi_{[n,n+1)}(h)h,$$

where χ_E is the characteristic function of a set E. By hypothesis there is $u_n \in U(M)$ such that $u_n h_n u_n^* = \alpha^{-1}(h_n)$, $n \in \mathbb{N}\cup\{0\}$. Let

$$v_n = \alpha^{-1}(\chi_{[n,n+1)}(h))u_n\,\chi_{[n,n+1)}(h),$$

and let $u = \sum_{n=0}^{\infty} v_n$. Then an easy computation shows u is unitary and $uhu^* = \alpha^{-1}(h)$. By assumption on α it is clear that τ is α-invariant on the ideal generated by $M^+ \cap L^1(M,\tau)$. Hence, if $y \in M^+$ then

$$\phi o \alpha(y) = \tau(h\alpha(y)) = \sum \tau(h_n\alpha(y)) = \sum \tau(\alpha^{-1}(h_n)y)$$
$$= \tau(\alpha^{-1}(h)y) = \tau(uhu^*y) = \tau(hu^*yu)$$
$$= \phi(u^*yu)$$

proving that α is pointwise inner. QED

Recall that a faithful normal state or weight ϕ is called lacunary if 1 is an isolated point of the spectrum $Sp(\Delta_\phi)$ of the modular operator Δ_ϕ, i.e. there exists $\lambda \in (0,1)$ such that

(1) $$Sp(\Delta_\phi) \cap (\lambda,\lambda^{-1}) = \{1\}.$$

It is a folklore result that a faithful normal semifinite lacunary weight is strictly semifinite. Since the result does not seem to exist is the literature, and it will make the rest of our discussion look nicer, we include a proof.

Lemma 2.3. Let ϕ be a faithful normal semifinite lacunary weight on M. Then ϕ is strictly semifinite.

Proof. Choose $\lambda \in (0,1)$ such that

$$Sp(\Delta_\phi) \cap (\lambda, \tfrac{1}{\lambda}) = \{1\}.$$

Put $\alpha = \log(\frac{1}{\lambda})$. Since the Arveson spectrum of the automorphism group $(\sigma_t^\phi)t\in\mathbb{R}$ is

$$\log(Sp(\Delta_\phi) \quad \{0\}),$$

cf. [2,Lem.3.2.2] it follows that if $f \in L^1(\mathbb{R})$ and supp $(\hat{f}) \subset (-\alpha,\alpha)$ then for all $x \in M$,

$$\int_{-\infty}^{\infty} \sigma_t^\phi(x)f(t)\,dt$$

is in the spectral subspace of M corresponding to $\{0\}$, i.e. it is in the fixed point algebra M_ϕ of σ^ϕ. We may choose f to be a positive continuous function such that $\int_{-\infty}^{\infty} f(t)dt = 1$, and $\text{supp}(\hat{f}) \subset (-\alpha,\alpha)$. Put

$$E(x) = \int_{-\infty}^{\infty} \sigma_t^\phi(x)f(t)dt.$$

Then by the above remarks E is a normal projection of M onto M_ϕ. Morever E is positive and $E(axb) = aE(x)b$, $a,b \in M_\phi$, $x \in M$, so that E is a conditional expectation of M onto M_ϕ.
Since ϕ is normal we can choose an increasing net $\{\phi_i\}_{i\in J}$ of positive normal functionals on M such that

$$\phi(x) = \sup_{i\in J} \phi_i(x) , \quad x \in M^+.$$

Hence for $x \in M^+$

$$\phi\circ E(x) = \sup_i \phi_i\left(\int_{-\infty}^{\infty} \sigma_t^\phi(x)f(t)dt\right)$$

$$= \sup_i \int_{-\infty}^{\infty} \phi_i(\sigma_t^\phi(x))f(t)dt$$

$$= \int_{-\infty}^{\infty} \phi(\sigma_t^\phi(x))f(t)dt$$

$$= \int_{-\infty}^{\infty} \phi(x)f(t)dt$$

$$= \phi(x).$$

Note that sup and $\int$ can be exchanged also if J is uncountable, because $t \to \phi_i(\sigma_t^\phi(x))f(t)$ form an increasing family of continuous functions on $\mathbb{R}$. Since $\phi\circ E = \phi$ it follows from [2,3.1.4] that ϕ is strictly semifinite. QED

In the rest of this section we shall show that the conclusion of Lemma 2.1 holds true for ϕ lacunary whenever α is approximately pointwise inner. If $\phi \in M_*^+$ is faithful we denote by $\| \ \|_\phi$ the norm $\|x\|_\phi = \phi(x^*x)^{\frac{1}{2}}$.

Lemma 2.4. Suppose ϕ is a faithful normal lacunary state on M. Let $u \in U(M)$. Then there exists $v \in U(M_\phi)$ such that

$$\|v-u\|_\phi \leq K_\lambda \|u\phi u - \phi\|^{1/4} ,$$

where $K_\lambda = \left(\frac{6}{1-\lambda^{\frac{1}{2}}}\right)^{\frac{1}{2}}$, and λ satisfies (1).

Proof. By Araki's generalization of an inequality of Powers and the second author [1] we have

$$\|uJ_\phi uJ_\phi\xi_\phi - \xi_\phi\| \leq \|u\phi u^* - \phi\|^{\frac{1}{2}} ,$$

where we have represented M in the GNS-representation due to ϕ, and J_ϕ is the conjugation such that $x^*\xi_\phi = J_\phi\Delta_\phi^{\frac{1}{2}}x\xi_\phi$, $x \in M$. The left side can be written as

$$\| u\xi_\phi - J_\phi u^* J_\phi \xi_\phi \| = \| (1-\Delta_\phi^{\frac{1}{2}})u\xi_\phi \| .$$

By assumption on $Sp(\Delta_\phi)$ we have for $t \in Sp(\Delta_\phi)-\{1\}$,

$$|t^{\frac{1}{2}}-1| \geq \min\{1-\lambda^{\frac{1}{2}}, \lambda^{-\frac{1}{2}}-1\} = 1-\lambda^{\frac{1}{2}}.$$

Choose a continuous real function f on $\mathbb{R}^+$ such that $f(1) = 1$, $f(t) = 0$ for $t \in Sp(\Delta_\phi)-\{1\}$. By spectral theory the operator $P_\phi = f(\Delta_\phi)$ is the projection on the eigenspace 1 for Δ_ϕ, and by the above inequality we have

$$|f(t)-1|^2 \leq \frac{1}{(1-\lambda^{\frac{1}{2}})^2} |t^{\frac{1}{2}}-1|^2, \ t \in Sp(\Delta_\phi).$$

It follows that

$$\begin{aligned} \| (f(\Delta_\phi)-1)u\xi_\phi \|^2 &\leq \frac{1}{(1-\lambda^{\frac{1}{2}})^2} \| (\Delta_\phi^{\frac{1}{2}}-1)u\xi_\phi \|^2 \\ &\leq \frac{1}{(1-\lambda^{\frac{1}{2}})^2} \| u\phi u^* - \phi \| . \end{aligned}$$

Let E_ϕ denote the ϕ-invariant normal coonditional expectation of M onto M_ϕ. Then we have

$$P_\phi(x\xi_\phi) = E_\phi(x)\xi_\phi , \quad x \in M ,$$

see e.g. [5, Thm. 1]. It thus follows that

$$\| (E_\phi(u)-u)\xi_\phi \|^2 \leq \frac{1}{(1-\lambda^{\frac{1}{2}})^2} \| u\phi u^* - \phi \| .$$

Put $\varepsilon = \| (E_\phi(u)-u)\xi_\phi \|$. Since $\| 1-P_\phi \| \leq 1$ we have $\varepsilon \leq 1$. Now M_ϕ is a finite von Neumann algebra. Hence there is $v \in U(M_\phi)$ such that the polar decomposition for $E_\phi(u)$ is given by

$$E_\phi(u) = v|E_\phi(u)| .$$

Put $h = |E_\phi(u)|$. Then $0 \leq h \leq 1$, so that $h^2 + (1-h)^2 \leq 1$, whence

$$\| (1-h)\xi_\phi \|^2 \leq 1 - \| h\xi_\phi \|^2 = 1 - \| E_\phi(u)\xi_\phi \|^2 \leq 1-(1-\varepsilon)^2 \leq 2\varepsilon .$$

It follows that

$$\begin{aligned}\| v\xi_\phi - u\xi_\phi \| &\leq \| v\xi_\phi - E_\phi(u)\xi_\phi\| + \varepsilon \\ &= \| v(h-1)\xi_\phi\| + \varepsilon \\ &\leq (2\varepsilon)^{\frac{1}{2}} + \varepsilon \\ &< (6\varepsilon)^{\frac{1}{2}}.\end{aligned}$$

Since $\varepsilon \leq \frac{1}{(1-\lambda^{\frac{1}{2}})} \| u\phi u^* - \phi\|^{\frac{1}{2}}$ we conclude that

$$\| v-u\|_\phi \leq \left(\frac{6}{1-\lambda^{\frac{1}{2}}}\right)^{\frac{1}{2}} \| u\phi u^* - \phi\|^{1/4}.$$

QED

We recall from [7] that if $\phi,\psi \in M_*^+$ then $\phi\sim\psi$ if there is a sequence $(u_n) \subset U(M)$ such that $\lim_{n\to\infty}\| \phi - u_n\psi u_n^*\| = 0$.

<u>Lemma 2.5</u>. Let ϕ and ψ be faithful normal states on M such that $\phi\sim\psi$. If ϕ is lacunary then there is $u \in U(M)$ such that $u\phi u^* = \psi$.

<u>Proof</u>. Choose $\lambda \in (0,1)$ such that $Sp(\Delta_\phi)\cap(\lambda,\lambda^{-1}) = \{1\}$. Choose $u_n \in U(M)$ such that

$$\| u_n\phi u_n^* - \psi\| < 2^{-4n}.$$

Put $\phi_n = u_n\phi u_n^*$ and $v_n = u_{n+1}u_n^*$, $n \in \mathbb{N}$. Then

$$v_n\phi_n v_n^* = \phi_{n+1}.$$

Furthermore, since $\| \phi_{n+1} - \phi_n\| < 2^{-4n} + 2^{-4(n+1)} < 2\cdot 2^{-4n}$, we have

$$\| v_n\phi_n v_n^* - \phi_n\| < 2\cdot 2^{-4n}.$$

Put $K_\lambda = \left(\frac{6}{(1-\lambda^{\frac{1}{2}})}\right)^{\frac{1}{2}}$ as in Lemma 2.4. Since $Sp(\Delta_{\phi_n}) = Sp(\Delta_\phi)$ there exists by Lemma 2.4 $w_n \in U(M_{\phi_n})$ such that

$$\| w_n - v_n\|_{\phi_n} < 2^{1/4} K_\lambda\, 2^{-n}.$$

Put $s_n = v_n w_n^* \in U(M)$. Since $w_n \in U(M_{\phi_n})$ we have

$$\|s_n - 1\|_{\phi_n} = \|v_n - w_n\|_{\phi_n} < 2^{1/4} K_\lambda 2^{-n}.$$

Also since $w_n \in U(M_{\phi_n})$,

$$s_n \phi_n s_n^* = v_n \phi_n v_n^* = \phi_{n+1}, \qquad n \in \mathbb{N}.$$

Define recursively $t_n \in U(M)$ by

$$t_1 = u_1$$
$$t_n = s_{n-1} \cdots s_1 u_1, \qquad n \geq 2.$$

Then we have by induction

$$t_n \phi t_n^* = \phi_n.$$

Using this we have

$$\begin{aligned}
\|t_{n+1} - t_n\|_\phi &= \|(s_n - 1)t_n \xi_\phi\| \\
&= \phi(t_n^*(s_n - 1)^*(s_n - 1)t_n) \\
&= \phi_n((s_n - 1)^*(s_n - 1)) \\
&= \|s_n - 1\|_{\phi_n} \\
&< {}^{1/4} K_\lambda 2^{-n}.
\end{aligned}$$

Since ϕ is faithful, (t_n) is a Cauchy sequence in the strong topology on the unit ball in M, hence there is an isometry $u \in M$ such that $t_n \to u$ strongly. For $x \in M$ we have

$$\begin{aligned}
\phi(x) &= \lim_{n\to\infty} \phi_n(x) \\
&= \lim_{n\to\infty} \phi(t_n^* x t_n) \\
&= \lim_{n\to\infty} (x t_n \xi_\phi, t_n \xi_\phi) \\
&= (x u \xi_\phi, u \xi_\phi) \\
&= \phi(u^* x u),
\end{aligned}$$

i.e. $\psi = u\phi u^*$. Since ψ is faithful and $u^*u = 1$, we have $1 \geq \psi(uu^*) = \phi(u^*uu^*u) = \phi(1) = 1$, so that $uu^* = 1$, whence u is unitary.

QED

Proposition 2.6. Suppose $\alpha \in \mathrm{Aut}(M)$ is approximately pointwise inner. Suppose ϕ is a faithful normal semifinite lacunary weight on M. Then there is $u \in U(M)$ such that

$$\phi o \alpha = u\phi u^*.$$

Proof. If ϕ is bounded we may assume ϕ is a state, and the proposition follows from Lemma 2.5.

Suppose ϕ is unbounded. Since ϕ is strictly semifinite by Lemma 2.3 there is a family $(p_i)_{i\in I}$ of orthogonal projections in M_ϕ with sum 1 such that $\phi(p_i) < \infty$ for all i. Notice that each p_i is σ-finite having finite value under ϕ. Since α is approximately pointwise inner, if $\psi \in M_*^+$, then $\psi o \alpha \sim \psi$ [7, 12.3 (4)], whence the support projections $\mathrm{supp}(\psi o \alpha) \sim \mathrm{supp}(\psi)$ in M by [7,Thm.2.2]. In particular $\alpha(p_i) \sim p_i$ for all i. Thus if we compose α by an inner automorphism, we may assume $\alpha(p_i) = p_i$, $i \in I$, whence α restricts to an automorphism of $p_i M p_i$ which is also approximately pointwise inner [7,Thm.2.2]. Furthermore we have

$$\mathrm{Sp}(\Delta_{\phi_{p_i}}) \cap (\lambda, \lambda^{-1}) = \{1\},$$

hence by Lemma 2.5 there exists $v_i \in M$ with $v_i^* v_i = v_i v_i^* = p_i$ such that

$$v_i \phi_{p_i} v_i^* = (\phi o \alpha)_{p_i}, \qquad i \in I,$$

where we have used that $p_i \in M_{\phi o \alpha}$, since $\alpha(p_i) = p_i \in M_\phi$. Put $v = \sum_{i\in I} v_i$. Then $v \in U(M)$ and $v\phi v^* = \phi o \alpha$.

QED

3 Factors of type III_λ, $0<\lambda<1$, with separable predual

Throughout this section M will denote a factor of type III_λ, $0<\lambda<1$, with separable predual. By [2,Thm.4.3.2 and Lem.5.3.2] there exists a faithful normal strictly semifinite lacunary weight ϕ with infinite multiplicity on M. Then M_ϕ is of type II_∞. By [2,p.238] there is a unitary operator $U \in M(\sigma^\phi,(1,\infty))$ such that $UM_\phi U^* = M_\phi$, and M_ϕ and U together generate M. Indeed M can be identified with the crossed product $M_\phi \times_\theta Z$, where $\theta = \mathrm{Ad}\, U|_{M_\phi}$. Moreover, by [2,p.241] U is unique modulo M_ϕ, and there is a unique element ρ of the center C_ϕ of M_ϕ such that $U^*\phi U = \phi(\rho\,.)$. Furthermore $0<\rho<\lambda_0<1$ for some $\lambda_0 \in \mathbb{R}$. Using this notation we have:

<u>Theorem 3.1</u>. Let $\alpha \in \mathrm{Aut}(M)$. Then α is pointwise inner if and only if there is $v \in U(M)$ such that ϕ is $\alpha \circ \mathrm{Ad}v$-invariant and $\alpha \circ \mathrm{Ad}v|_{M_\phi} = \iota$ - the identity map.

<u>Proof</u>. Assume α is pointwise inner. By Lemma 2.1 there is $u \in U(M)$ such that $\phi \circ \alpha = u\phi u^*$. Thus if we replace α by $\alpha \circ \mathrm{Ad}u$ we may, and do, assume $\phi \circ \alpha = \phi$. Let $h \in M_\phi^+$ with $\phi(h) < \infty$. Choose positive real numbers a, b such that $k = ah + b1$ satisfies $\rho<\lambda_0<k<1$. If we can show there is $u \in U(M)$ such that $\alpha(k) = u^*ku$ then also $\alpha(h) = u^*hu$. Since ϕ is α-invariant, $\alpha(M_\phi) = M_\phi$, i.e. $\alpha|_{M_\phi} \in \mathrm{Aut}(M_\phi)$. Since the commutant of k in M_ϕ equals the commutant of h in M_ϕ it follows that the weight $\phi(k.)$ is strictly semifinite. Thus by Lemma 2.1 there is $u \in U(M)$ such that $\phi(k\alpha^{-1}(x)) = \phi(kuxu^*)$, $x \in M^+$, or by the α-invariance of ϕ, $\phi(\alpha(k)x) = \phi(kuxu^*)$, $x \in M^+$. Since $\rho<\lambda_0 1<k<1$, $\rho<\alpha(k)<1$ as well. Then by [4,Lem.I.2.6.(c)] applied to the weights $\phi(k.)$ and $\phi(\alpha(k).)$ we have $u \in M_\phi$. Thus $\phi(\alpha(k).) = \phi(u^*ku.)$. In particular

this holds for the restriction to M_ϕ, hence $\alpha(k) = u^*ku$, and so $\alpha(h) = u^*hu$. By Lemma 2.2 $\alpha|_{M_\phi}$ is pointwise inner, whence $\alpha|_{M_\phi}$ is inner by [7,Prop.12.5]. Let $v \in U(M_\phi)$ satisfy $\alpha|_{M_\phi} = \mathrm{Ad}v^*|_{M_\phi}$. Then $\alpha o \mathrm{Ad}v|_{M_\phi} = \iota$, proving the necessity.

To prove the converse we may assume $\alpha|_{M_\phi} = \iota$ and $\phi o \alpha = \phi$. Let E be the unique bounded ϕ-invariant faithful normal conditional expectation of M onto M_ϕ. Let $\psi \in M_*^+$. Assume first that there is $h \,\eta\, M_\phi^+$ such that $\psi(x) = \phi(hE(x))$, $x \in M$. Since $\alpha|_{M_\phi} = \iota$, $E = \alpha o E$, and by uniqueness of E, $\alpha o E = E o \alpha$, whence $\psi o \alpha = \psi$. In the general case there are by [4,Thm.I.2.2] $h \in M_\phi^+$ and $w \in U(M)$ such that $\psi o \mathrm{Ad}w(x) = \phi(hE(x))$, $x \in E$. By the above $\psi o \mathrm{Ad}w o \alpha = \psi o \mathrm{Ad}w$. Since $\mathrm{Ad}w o \alpha = \alpha o \mathrm{Ad}\alpha^{-1}(w)$ we have

$$\psi \; o \; \alpha \; o \; \mathrm{Ad}\; \alpha^{-1}(w) = \psi \; o \; \mathrm{Ad}\; w$$

or $\psi o \alpha = \psi o \mathrm{Ad}(w\alpha^{-1}(w^*))$. Since ψ was arbitrary in M_*^+, α is pointwise inner.

QED

We denote by $\mathrm{Aut}_\phi(M)$ the subgroup of $\mathrm{Aut}(M)$,

$$\mathrm{Aut}_\phi(M) = \{\alpha \in \mathrm{Aut}(M): \; \phi o \alpha = \phi, \alpha|_{M_\phi} = \iota\}.$$

By Theorem 3.1 $\mathrm{Aut}_\phi(M)$ is a subgroup of the group of pointwise inner automorphisms.

<u>Theorem 3.2</u>. (1) The map $\alpha \to w_\alpha = \alpha(U)U^*$ is an isomorphism of $\mathrm{Aut}_\phi(M)$ onto $U(C_\phi)$.

(2) If $\alpha \in \mathrm{Aut}_\phi(M)$ then α is inner if and only if w_α is a θ-coboundary, i.e. there is $v \in U(C_\phi)$ such that $w_\alpha = v\theta(v)^*$, where $\theta = \mathrm{Ad}U|_{M_\phi}$.

<u>Proof</u>. (1) Let $\alpha \in \mathrm{Aut}_\phi(M)$ and $x \in M_\phi$. Then

$$\begin{aligned} w_\alpha x &= \alpha(U)U^* x = \alpha(U)U^* xUU^* = \alpha(U(U^* xU))U^* \\ &= x\, \alpha(U)U^* = x\, w_\alpha . \end{aligned}$$

Thus $w_\alpha \in M_\phi' \cap M$, which is equal to C_ϕ by [2,lem.4.2.3] and [4,Cor.I.2.10]. Thus $w_\alpha \in U(C_\phi)$. If $\alpha, \beta \in \mathrm{Aut}_\phi(M)$ then

$$w_{\alpha\beta} = \alpha(\beta(U))U^* = \alpha(w_\beta U)U^* = w_\beta \alpha(U)U^* = w_\beta w_\alpha = w_\alpha w_\beta,$$

whence the map $\alpha \to w_\alpha$ is a homomorphism. If $w_\alpha = w_\beta$ then $\alpha(U) = \beta(U)$. Since $\alpha|_{M_\phi} = \beta|_{M_\phi} = \iota$, and U and M_ϕ generate M, $\alpha = \beta$. Thus the map is injective.

To show surjectivity let $w \in U(C_\phi)$. Let $w_1 = w$, and for $n \in \mathbb{Z}$ define w_n by the formula

$$w_n = \begin{cases} w_{n-1}\theta^{n-1}(w), & n \geq 2 \\ w_{n+1}\theta^n(w)^*, & n \leq 0. \end{cases}$$

Then $n \to w_n$ is a θ-cocycle, and we have the formulas:

(1) $$w_{m+n} = w_n \theta^n(w_m), \qquad n,m \in \mathbb{Z}$$

(2) $$w_n = w\,\theta(w)\dots\theta^{n-1}(w), \qquad n \in \mathbb{N}$$

(3) $$w_{-n} = \theta^{-1}(w)^*\theta^{-2}(w)^*\dots\theta^{-n}(w)^*, \qquad n \in \mathbb{N}.$$

Let A denote the $*$-algebra of finite sums $\sum x_n U^n$, $x_n \in M_\phi$ and define a map $\alpha_w : A \to A$ by

$$\alpha_w(\sum x_n U^n) = \sum x_n w_n U^n.$$

From (1) it is easy to show α_w is multiplicative. From (2) and (3) we obtain $\theta^n(w_{-n})^* = w_n$ or $w_{-n} = \theta^{-n}(w_n)$, from which it follows that α_w is $*$-preserving and thus a $*$-automorphism of A.

Let $h \in M_\phi^+$ be an operator such that $\omega = \phi(h.)$ is a faithful normal state on M. Then $\omega(xU^n) = 0$ for $x \in M_\phi$ and $n \neq 0$, hence $\omega|_A$ is α_w-invariant. Thus α_w is unitarily implemented in the GNS-representation π_ω of $\omega|_A$, hence α_w extends by continuity to the weak closure. Since ω is normal and faithful π_ω extends to an isomorphism of M onto $\pi_\omega(A)^-$. Thus α_w extends to an automorphism α of M. By construction $\alpha \in \mathrm{Aut}_\phi(M)$ and $w_\alpha = w$. Thus $\alpha \to w_\alpha$ is surjective, hence is an isomorphism of $\mathrm{Aut}_\phi(M)$ onto $U(C_\phi)$.

(2) Suppose $w_\alpha = v\theta(v)^*$ is a coboundary with $v \in U(C_\phi)$. Then $\mathrm{Adv}(U) = vUv^* = v\theta(v)^* U = w_\alpha U$, and $\mathrm{Adv}|_{M_\phi} = \iota$. Thus $\mathrm{Adv} \in \mathrm{Aut}_\phi(M)$ and is equal to α by uniqueness, whence α is inner.

Conversely, if $\alpha = \mathrm{Adv}$, $v \in U(M)$ then $\phi o \alpha = \phi$ implies $v \in M_\phi$ and since $\alpha|_{M_\phi} = \iota$, $v \in M_\phi' \cap M_\phi = C_\phi$. Thus $w_\alpha = \alpha(U)U^* = v\theta(v)^*$ with $v \in U(C_\phi)$.

QED

Recall from [7,Prop.12.6] that each modular automorphism is pointwise inner.

<u>Corollary 3.3</u>. Let M be of type III_λ, $0<\lambda<1$, with separable predual and let $\alpha \in \mathrm{Aut}(M)$. Then α is pointwise inner if and only if there are $t \in \mathbb{R}$ and $u \in U(M)$ such that $\alpha = \sigma_t^\phi o\ \mathrm{Adu}$.

<u>Proof</u>. Let ϕ be a generalized trace of M [2,Def.4.3.1]. Then $C_\phi = \mathbb{C}1$, hence $U(C_\phi) = \mathbb{T}$, and so $\mathrm{Aut}_\phi(M) \simeq \mathbb{T}$, and the only θ-coboundary is 1. Since M is of type III_λ, σ_t^ϕ is outer for $t \notin \frac{2\pi}{\log\lambda}\mathbb{Z}$. Thus if $w_t = \sigma_t^\phi(U)U^*$, then $w_t \neq 1$ if and only if $t \notin \frac{2\pi}{\log\lambda}\mathbb{Z}$. Since σ_t^ϕ is continuous and periodic, the range of $t \to w_t$ is the whole circle $\mathbb{T}$. It then follows from Theorem 3.2 that $\alpha \in \mathrm{Aut}_\phi(M)$ if and only if $\alpha = \sigma_t^\phi$ for some t, and hence by Theorem 3.1 that α is pointwise inner if and only if $\alpha = \sigma_t^\phi o\ \mathrm{Adu}$ for some $t \in \mathbb{R}$, $u \in U(M)$.

QED

4 Cohomology

We show in the present section that the pointwise inner automorphisms modulo the inner can be described as a cohomology group $H^1(\mathbb{Z},U(C_\phi))$. We retain the notation introduced in section 3. As remarked in the proof of Theorem 3.2 each $w \in U(C_\phi)$ defines a θ-cocycle, and conversely each θ-cocycle $(w_n)_{n\in\mathbb{Z}}$ is uniquely determined by w_1. Since $U(C_\phi)$ is an abelian group, this association defines a multiplicative isomorphism

$$\gamma: U(C_\phi) \to Z^1(\mathbb{Z},U(C_\phi))$$

onto the multiplicative group of θ-cocycles in $U(C_\phi)$.
Let $B^1(\mathbb{Z},U(C_\phi))$ be the image under γ of the θ-coboundaries $v\theta(v)^*$, $v \in U(C_\phi)$. By Theorem 3.2 $B^1(\mathbb{Z},U(C_\phi))$ is the image in $Z^1(\mathbb{Z},U(C_\phi))$ of the inner automorphisms in $\mathrm{Aut}_\phi(M)$ under the composition $\alpha \to w_\alpha \to \gamma(w_\alpha)$. Let $H^1(\mathbb{Z},U(C_\phi))$ be the cohomology group

$$H^1(\mathbb{Z},U(C_\phi)) = Z^1(\mathbb{Z},U(C_\phi))/B^1(\mathbb{Z},U(C_\phi)),$$

and let ε be the canonical homomorphism

$$\varepsilon: \mathrm{Aut}(M) \to \mathrm{Aut}(M)/\mathrm{Int}(M) = \mathrm{Out}(M),$$

where $\mathrm{Int}(M)$ denotes the inner automorphisms. Summarizing we have proved:

<u>Lemma 4.1</u>. There is a natural isomorphism

$$\varepsilon(\mathrm{Aut}_\phi(M)) \simeq H^1(\mathbb{Z},U(C_\phi))$$

induced by the composition $\alpha \to w_\alpha \to \gamma(w_\alpha)$.

Denote by

$$\mathrm{Pt\ Int}(M) = \{\alpha\in\mathrm{Aut}(M): \alpha \text{ is pointwise inner}\}$$

By Theorem 3.1

$$\varepsilon(\mathrm{Pt\ Int}(M)) = \varepsilon(\mathrm{Aut}_\phi(M))$$

Since $\varepsilon(\mathrm{Pt\ Int}(M))$ is independent of the weight ϕ we have by use of Lemma 4.1:

Proposition 4.2. The cohomology group $H^1(\mathbb{Z},U(C_\phi))$ is independent of the lacunary weight ϕ, and we have an isomorphism

$$\varepsilon(\mathrm{Pt\ Int}(M)) \simeq H^1(\mathbb{Z},U(C_\phi))$$

If we apply this to the generalized trace in a III_λ-factor we obtain from the proof of Corollary 3.3.

Corollary 4.3. If M is of type III_λ, $0<\lambda<1$, then

$$H^1(\mathbb{Z},U(C_\phi)) \simeq \mathbb{T}$$

If M is of type III_0 we can give an alternative description of $\varepsilon(\mathrm{Pt\ Int}(M))$. We denote by

$$\mathrm{Int}_{C_\phi}(M) = \{\mathrm{Ad}w \in \mathrm{Aut}(M) : w \in U(C_\phi)\}$$

We denote by $\overline{\mathrm{Int}_{C_\phi}(M)}$ the closure of $\mathrm{Int}_{C_\phi}(M)$ in $\mathrm{Aut}(M)$, where $\mathrm{Aut}(M)$ has the topology of pointwise norm convergence in M.

Theorem 4.4. Suppose M is of type III_0, and let $\alpha \in \mathrm{Aut}(M)$. Then $\alpha \in \mathrm{Aut}_\phi(M)$ if and only if $\alpha \in \overline{\mathrm{Int}_{C_\phi}(M)}$.

Proof. Suppose $\alpha \in \mathrm{Aut}_\phi(M)$. Since M is of type III_0, C_ϕ is purely nonatomic hence isomorphic to $L^\infty([0,1],\mu)$ for a nonatomic measure μ, and θ corresponds to an ergodic nonsingular transformation leaving μ quasiinvariant. By the alternate form of the Rokhlin lemma, see [12,p.12], there is a sequence (v_n) in $U(C_\phi)$ such that $v_n\theta(v_n)^* \to w_\alpha$ strongly. The strong convergence follows since in the Rokhlin lemma the convergence in L^∞ is the one induced by L^1-convergence. Thus $\mathrm{Ad}v_n(U) \to \alpha(U)$ strongly. Since M is generated by sums $\sum_{k\in\mathbb{Z}} x_k U^k$, $x_k \in M_\phi$, and $\mathrm{Ad}v_n|_{M_\phi} = \alpha|_{M_\phi} = \iota$, it follows that $\alpha = \lim_n \mathrm{Ad}v_n$ in $\mathrm{Aut}(M)$, i.e. $\alpha \in \overline{\mathrm{Int}_{C_\phi}(M)}$.

Conversely suppose $\alpha \in \overline{\mathrm{Int}(C_\phi)}$. Let (v_n) be a sequence in $U(C_\phi)$ such that $\mathrm{Ad}v_n \to \alpha$ in $\mathrm{Aut}(M)$. If $x \in M_\phi$ then $\alpha(x) = \lim_n v_n x v_n^*$ $= x$, so $\alpha|_{M_\phi} = \iota$. But then as in the proof of Theorem 3.2 $w_\alpha =$ $\alpha(U)U^* \in U(C_\phi)$, hence $\alpha(U^n) = w_n U^n$ with $w_n \in U(C_\phi)$, $n \in \mathbf{Z}$, hence ϕ is α-invariant and $\alpha \in \mathrm{Aut}_\phi(M)$.

QED

If we combine this result with Lemma 4.1 and Proposition 4.2 we have

<u>Corollary 4.5</u>. Suppose M is of type III_0. Then $\overline{\mathrm{Int}_{C_\phi}(M)}/\mathrm{Int}_{C_\phi}(M)$ is independent of the weight ϕ, and we have an isomorphism

$$\varepsilon(\mathrm{Pt\ Int}(M)) \simeq \overline{\mathrm{Int}_{C_\phi}(M)}/\mathrm{Int}_{C_\phi}(M).$$

If we combine Theorems 3.1 and 4.4 we find:

<u>Corollary 4.6</u>. Suppose M is of type III_0. Then every pointwise inner automorphism of M is approximately inner.

<u>Remark 4.7</u>. Corollary 4.6 is an extension of a result of Connes [3,Prop.3.9] in which he showed that all modular automorphisms of M are approximately inner. Indeed by [7,Prop.12.6] all modular automorphisms are pointwise inner.

<u>Remark 4.8</u>. In the appendix C. Sutherland will show that in its natural Borel structure $H^1(\mathbf{Z}, U(C_\phi))$ is a nonsmooth Borel space. This can be used to show that in at least some cases there exist pointwise inner automorphisms which are not of the form $\sigma_t^\phi \circ \mathrm{Ad}u$. Indeed, let M be of type III_0 with $T(M)$ a closed subgroup of $\mathbf{R}$. Then $\varepsilon(\{\sigma_t^\phi : t \in \mathbf{R}\}) \simeq \mathbf{R}/T(M)$, which is smooth by the assumption on $T(M)$, hence $\varepsilon(\{\sigma_t^\phi : t \in \mathbf{R}\}) \neq \varepsilon(\mathrm{Pt\ Int}(M))$.

5 The continuous crossed product

Let M be a von Neumann algebra acting on a Hilbert space H. Let ω be a faithful normal semifinite weight on M with modular group σ^ω. Then the crossed product $N = M \times_{\sigma^\omega} \mathbb{R}$ is the von Neumann algebra acting on $L^2(\mathbb{R},H)$ generated by operators $\acute{\pi}(x)$, $x \in M$, and $\lambda(t)$, $t \in \mathbb{R}$, defined as follows

$$(\acute{\pi}(x)\xi)(s) = \sigma^\omega_{-s}(x)\xi(s) \ , \qquad \xi \in L^2(\mathbb{R},H)$$

$$(\lambda(t)\xi)(s) = \xi(s-t) \ , \qquad s \in \mathbb{R}.$$

$\acute{\pi}$ is a representation of M into N and λ is a unitary representation of $\mathbb{R}$ in N implementing σ^ω. The dual automorphism group θ of σ^ω on N is the automorphism group determined by

$$\theta_s(\acute{\pi}(x)) = \acute{\pi}(x) \ , \qquad x \in M$$

$$\theta_s(\lambda(t)) = e^{-ist}\,\lambda(t) \ , \qquad s,t \in \mathbb{R}.$$

Then $\acute{\pi}(M)$ is the fixed point algebra of θ. By [6] there is a faithful normal semifinite operator valued weight T of N on $\acute{\pi}(M)$ given by

$$T(y) = \int_{-\infty}^{\infty} \theta_s(y)\, ds,$$

where ds denotes the Lebesgue measure on $\mathbb{R}$. Then for any normal semifinite weight ϕ on M its dual weight $\tilde{\phi}$ on N is given by

$$\tilde{\phi} = \phi \circ \acute{\pi}^{-1} \circ T.$$

By [14,Lem.8.2] there is a positive self-adjoint operator h affiliated with N such that $\lambda(t) = h^{it}$, and the weight τ defined by

$$\tau(y) = \tilde{\omega}(h^{-1}y) \ , \qquad y \in N^+,$$

is a faithful normal semifinite trace on N such that

$$\tau \circ \theta_s = e^{-s}\,\tau \ , \qquad s \in \mathbb{R}.$$

τ is called the canonical trace on N.

By [14,Props.3.5,4.2] this construction is independent of the weight ω up to isomorphism. Let $\alpha \in \mathrm{Aut}(M)$. By [7,Prop.12.1] there exists a unique automorphism $\tilde{\alpha} \in \mathrm{Aut}(N)$ such that

$$\tilde{\alpha}(\acute{\pi}(x)) = \acute{\pi}(\alpha(x)), \qquad x \in M$$

$$\tilde{\alpha}(\lambda(s)) = \acute{\pi}((D_\omega o\alpha^{-1} : D(\omega))_s)\lambda(s), \qquad s \in \mathbb{R}.$$

We want to remark that $\tilde{\alpha}$ also has an abstract characterization.

Lemma 5.1. Let $\alpha \in \mathrm{Aut}(M)$. Then $\tilde{\alpha}$ is the unique automorphism of N such that

$$\tilde{\alpha}(\acute{\pi}(x)) = \acute{\pi}(\alpha(x)), \quad x \in M,$$

$$\tilde{\alpha} o \theta_s = \theta_s o \tilde{\alpha}, \quad s \in \mathbb{R},$$

$$\tau o \tilde{\alpha} = \tau.$$

Proof Suppose $\tilde{\alpha}_1$ and $\tilde{\alpha}_2$ both satisfy the three conditons of the lemma, $\alpha_1, \alpha_2 \in \mathrm{Aut}(M)$. Put $\beta = \tilde{\alpha}_2^{-1} o \tilde{\alpha}_1$. Then $\beta \in \mathrm{Aut}(N)$, and

$$\beta(\acute{\pi}(x)) = \acute{\pi}(x), \quad x \in M,$$

$$\beta o \theta_s = \theta_s o \beta, \quad s \in \mathbb{R},$$

$$\tau o \beta = \tau.$$

The second formula implies that β commutes with the operator valued weight T defined above. By the first formula $\acute{\pi}^{-1}(\beta(y)) = \acute{\pi}^{-1}(y)$ for $y \in \acute{\pi}(M)$, so that

$$\tilde{\omega} o \beta = \omega o \acute{\pi}^{-1} o T o \beta = \omega o \acute{\pi}^{-1} o T = \tilde{\omega}$$

Since also $\tau o \beta = \tau$, the Radon-Nikodym derivative $\frac{d\tilde{\omega}}{d\tau}$ is β-invariant. But $\lambda(t) = (\frac{d\tilde{\omega}}{d\tau})^{it}$. Hence β acts trivially on $\acute{\pi}(x)$, $x \in M$, and $\lambda(t)$, $t \in \mathbb{R}$, i.e. β is the identity on N, proving the uniqueness of $\tilde{\alpha}$. Q.E.D.

It was shown in [7,Thm.12.4] that α is approximately pointwise inner if and only if $\tilde{\alpha}|_{Z(N)} = \iota$, where $Z(N)$ denotes the center of N.

We shall in the present section be concerned with the case when α is pointwise inner. Notice that if $\alpha = \mathrm{Ad}u$ is inner then by Lemma 5.1

$$\tilde{\alpha} = \mathrm{Ad}\,\hat{\pi}(u) \in \mathrm{Int}(N).$$

We should remark that it follows from Lemma 5.1 that if ϕ and ω are faithful normal semifinite weights, and χ is the isomorphism of $M \times_{\sigma^\phi} \mathbb{R}$ onto $M \times_{\sigma^\omega} \mathbb{R}$ constructed in [14, Prop. 3.5] then χ carries the automorphism $\tilde{\alpha}$ defined with respect to $M \times_{\sigma^\phi} \mathbb{R}$ onto the one defined with respect to $M \times_{\sigma^\omega} \mathbb{R}$. Thus in order to show $\tilde{\alpha}$ is inner, it suffices to do this in $\mathrm{Aut}(M \times_{\sigma^\phi} \mathbb{R})$ for some suitably chosen ϕ.

<u>Theorem 5.2</u>. Suppose M is a factor of type III_λ, $0<\lambda<1$, with separable predual, and let $\alpha \in \mathrm{Aut}(M)$. Then α is pointwise inner if and only if $\tilde{\alpha}$ is inner on N.

We first prove a lemma.

<u>Lemma 5.3</u>. Let ϕ be a faithful normal semifinite weight on a von Neumann algebra M. Put $N = M \times_{\sigma^\phi} \mathbb{R}$ and let $\tilde{\phi}$ be the dual weight of ϕ on N. Then the centralizer $N_{\tilde{\phi}}$ of $\tilde{\phi}$ is generated by $\hat{\pi}(M_\phi)$ and $\lambda(\mathbb{R})$, i.e.

$$N_{\tilde{\phi}} \cong M_\phi \times_{\sigma^\phi} \mathbb{R} \cong M_\phi \hat{\otimes} L^\infty(\mathbb{R}).$$

<u>Proof</u>. It is clear, that the von Neumann algebra generated by $\hat{\pi}(M_\phi)$ and $\lambda(\mathbb{R})$ is contained in $N_{\tilde{\phi}}$. To prove the converse inclusion, note first that N is contained in $M \hat{\otimes} B(L^2(\mathbb{R}))$ with the usual identification of $L^2(\mathbb{R},H)$ and $H \otimes L^2(\mathbb{R})$. Let Tr be the trace on $B(L^2(\mathbb{R}))$ and put $\omega = \phi \otimes \mathrm{Tr}$. Then

$$\sigma_t^\omega = \sigma_t^\phi \otimes i_{B(L^2(\mathbb{R}))} .$$

One checks easily that

$$\sigma_t^\omega(\pi(x)) = \pi(\sigma_t^\omega(x)) \ , \quad x \in M ,$$

$$\sigma_t^\omega(\lambda(s)) = \lambda(s) \ , \quad s \in \mathbb{R} .$$

Hence σ_t^ω maps N into itself and the restriction of σ_t^ω to N coincides with $\sigma_t^{\tilde{\phi}}$. Since the centralizer of ω is $M_\phi \hat{\otimes} B(L^2(\mathbb{R}))$,

$$M_{\tilde{\phi}} = N \cap (M_\phi \hat{\otimes} B(L^2(\mathbb{R}))) .$$

Since also $\sigma_t^{\tilde{\phi}} = \mathrm{Ad}_N(\lambda(t))$ we have

$$M_{\tilde{\phi}} \subset (M_\phi \hat{\otimes} B(L^2(\mathbb{R}))) \cap \lambda(\mathbb{R})' .$$

But $\lambda(t) = 1 \otimes l(t)$, where $l(t)$ denotes the left translation by t on $L^2(\mathbb{R})$. The von Neumann algebra A generated by $l(\mathbb{R})$ is a maximal abelian subalgebra of $B(L^2(\mathbb{R}))$ isomorphic to $L^\infty(\mathbb{R})$. Hence

$$N_{\tilde{\phi}} \subset M_\phi \hat{\otimes} A' = M_\phi \hat{\otimes} A .$$

This completes the proof of lemma 5.3, because $M_\phi \hat{\otimes} A$ is the von Neumann algebra generated by $\pi(M_\phi)$ and $\lambda(\mathbb{R})$.

Q.E.D.

<u>Proof of Theorem 5.2.</u> We first assume α is pointwise inner.

Let ϕ and U be as above, and let $\beta = Ad\bar{\pi}(U) \in Aut(N)$. Then $\beta(\bar{\pi}(M_\phi)) = \bar{\pi}(M_\phi)$, and since by Theorem 3.2 $\sigma_t^\phi(U) = w_t U$, $w_t \in U(C_\phi)$,

$$\lambda(t)\bar{\pi}(U)\lambda(-t) = \bar{\pi}(w_t)\bar{\pi}(U) \quad \text{or} \quad \beta(\lambda(t)) = \bar{\pi}(w_t)^*\lambda(t),$$

it follows from Lemma 5.3 that $\beta(N_{\tilde{\phi}}) = N_{\tilde{\phi}}$. By [2,p.241] there is $\rho \in C_\phi$ such that $0 \leq \rho \leq \lambda_0 < 1$ for some $\lambda_0 \in \mathbb{R}$ and $\phi_U = \phi(\rho\cdot)$. Let $h_\phi = \frac{d\tilde{\phi}}{d\tau}$. Then if $x \in N^+ \cap L^1(N,\tau)$ we have

$$\begin{aligned}
\tau(\beta^{-1}(h_\phi)x) &= \tau(h_\phi\beta(x)) \\
&= \tilde{\phi}(\beta(x)) \\
&= \phi\circ\bar{\tau}^{-1}\left(\int_{-\infty}^{\infty} \theta_s(\beta(x))ds\right) \\
&= \phi(U\bar{\pi}^{-1}(T(x))U^*) \\
&= \phi(\rho\bar{\pi}^{-1}(T(x))) \\
&= \tau(h_\phi\bar{\pi}(\rho)x).
\end{aligned}$$

Note that by Lemma 5.3 $\bar{\pi}(\rho) \in Z(N_{\tilde{\phi}})$. Since h_ϕ is affiliated with $Z(N_{\tilde{\phi}})$ so is therefore $\bar{\pi}(\rho)h_\phi$. Since x above is arbitrary

$$\beta^{-1}(h_\phi) = h_\phi\bar{\pi}(\rho) \leq \lambda_0 h_\phi.$$

By [11,23.13] there is a projection $e \in Z(N_{\tilde{\phi}})$ such that $\sum_{n\in\mathbb{Z}} \beta^n(e) = 1$.

We assert that if $u \in U(Z(N_{\tilde{\phi}}))$ then u is a β-coboundary. Indeed, let $Z_n = e_n Z(N_{\tilde{\phi}})$, where $e_n = \beta^n(e)$, and let $b_n = e_n b$ for $b \in Z(N_{\tilde{\phi}})$. Put recursively

$$\begin{cases}
w_0 = 1 & \\
w_n = u_n\beta(w_{n-1}), & n \in \mathbb{N} \\
w_{-n} = \beta^{-1}(u_{-n+1}^*)\beta^{-1}(w_{-n+1}), & n \in \mathbb{N}.
\end{cases}$$

Since $\beta(Z_n) = Z_{n+1}$, $w_n \in Z_n$, $n \in \mathbb{Z}$. Computing we have for $w = \sum_{n\in\mathbb{Z}} w_n$,

$$\begin{aligned} w\beta(w)^* &= \sum_{n=1}^{\infty}(w_{-n}\beta(w_{-n-1})^*) + w_0\beta(w_{-1})^* + \sum_{n=1}^{\infty}(w_n\beta(w_{n-1})^*) \\ &= \sum_{n=1}^{\infty} w_{-n}u_{-n}w_{-n}^* + u_0 + \sum_{n=1}^{\infty} u_n\beta(w_{n-1})\beta(w_{n-1})^* \\ &= \sum_{n\in\mathbb{Z}} u_n = u \end{aligned}$$

proving the assertion.

Now consider $\alpha \in \mathrm{Aut}(M)$ which is pointwise inner. By Theorem 3.1 there is $v \in U(M)$ such that ϕ is $\alpha\circ\mathrm{Ad}v$-invariant and $\alpha\circ\mathrm{Ad}v|_{M_\phi} = \iota$. Since $(\mathrm{Ad}v)^\sim \in \mathrm{Int}(N)$ we may thus replace α by $\alpha\circ\mathrm{Ad}v$ and assume $\alpha \in \mathrm{Aut}_\phi(M)$. Let as in Theorem 3.2 $w_\alpha = \alpha(U)U^* \in U(C_\phi)$. By Lemma 5.3 $\bar{\pi}(w_\alpha) \in Z(N_{\tilde{\phi}})$, hence is a β-coboundary by the previous paragraph. Let $v \in U(Z(N_{\tilde{\phi}}))$ satisfy $\bar{\pi}(w_\alpha) = v\beta(v)^*$. Then we have

$$\begin{aligned} v\bar{\pi}(U)v^* &= v\bar{\pi}(U)v^*\bar{\pi}(U)^*\bar{\pi}(U) \\ &= v\beta(v^*)\bar{\pi}(U) = \bar{\pi}(w_\alpha U) \\ &= \tilde{\alpha}(\bar{\pi}(U)). \end{aligned}$$

Since $\mathrm{Ad}v|_{N_{\tilde{\phi}}} = \iota = \tilde{\alpha}|_{N_{\tilde{\phi}}}$, and N is generated by $N_{\tilde{\phi}}$ and $\bar{\pi}(U)$ by Lemma 5.3, $\tilde{\alpha} = \mathrm{Ad}v \in \mathrm{Int}(N)$.

Conversely assume $\tilde{\alpha} \in \mathrm{Int}(N)$. Let ϕ be a faithful normal strictly semifinite lacunary weight of infinite multiplicity on M. By [7,Thm.12.4] α is approximately pointwise inner, so by Proposition 2.6 there is $u \in U(M)$ such that $\phi\circ\alpha = u\phi u^*$. Again we may replace α by $\alpha\circ\mathrm{Ad}u$ and assume ϕ is α-invariant. Say $\tilde{\alpha} = \mathrm{Ad}v$, $v \in U(N)$. Since $\tilde{\phi} = \phi\circ\bar{\pi}^{-1}\circ T$, Lemma 5.1 shows $\tilde{\phi}$ is $\tilde{\alpha}$-invariant, hence $v \in N_{\tilde{\phi}}$. By Lemma 5.3 $N_\phi \simeq M_\phi \hat{\otimes} L^\infty(\mathbb{R},dx)$, hence we may write

$$N_{\tilde{\phi}} = \int_{\mathbb{R}}^{\oplus} M_\phi(t)\,dt$$

where $M_\phi(t) = M_\phi$. In particular

$$v = \int_{\mathbb{R}}^{\oplus} v_t\, dt \;, \qquad v_t \in M_\phi, \quad t \in \mathbb{R}.$$

Let $x \in M_\phi^+$. Then since $\bar{\pi}(x) = x \otimes 1$,

$$\bar{\pi}(x) = \int_{\mathbb{R}}^{\oplus} x_t\, dt \;, \qquad x_t = x, \quad t \in \mathbb{R}.$$

Hence we have

$$\bar{\pi}(\alpha(x)) = \tilde{\alpha}(\bar{\pi}(x)) = v\bar{\pi}(x)v^* = \int_{\mathbb{R}}^{\oplus} v_t x\, v_t^* dt.$$

Since $\alpha(x) \in M_\phi$, $\bar{\pi}(\alpha(x)) = \alpha(x)\otimes 1$, hence $v_t x\, v_t^* = \alpha(x)$ a.e. It follows that there is $w \in U(M_\phi)$ such that $wxw^* = \alpha(x)$. Since this holds for all $x \in M_\phi^+$, $\alpha|_{M_\phi}$ is pointwise inner by Lemma 2.2, and so by [7,Prop.12.5] $\alpha|_{M_\phi}$ is inner. Say $\alpha|_{M_\phi} = \mathrm{Ad}u|_{M_\phi}$, $u \in U(M_\phi)$. Then $\alpha \circ \mathrm{Ad}u \in \mathrm{Aut}_\phi(M)$, so α is pointwise inner by Theorem 3.1.

QED

For the following we refer to [4]. Let M be an infinite factor with separable predual. Let ω be a dominant weight, and $M = M_\omega \times_{\theta_0} \mathbb{R}$ be the continuous crossed product decomposition of M with respect to the centralizer M_ω and the one-parameter automorphism group θ_0 on M_ω such that $\omega \circ (\theta_0)_s = e^{-s}\omega$. Let $\{u(s)\}_{s\in\mathbb{R}}$ be the one-parameter unitary group in M implementing θ_0. By [4,Prop. IV.2.1, and Thms. IV.2.2 and 2.4] the extended modular automorphisms of M are up to multiples by inner automorphisms exactly the automorphisms $\bar{\sigma}_c^\omega$ such that $\bar{\sigma}_c^\omega|_{M_\omega} = \iota$. Here $c \in Z^1(F^M)$ - the continuous one-cocycles in the flow of weights with respect to the flow F^M.

If P is a von Neumann algebra and σ is a continuous representation of $\mathbb{R}$ in $\mathrm{Aut}(P)$ then an automorphism α of P which commutes with σ, extends to an automorphism $\tilde{\alpha}$ of $P \times_\sigma \mathbb{R}$ which leaves fixed the unitaries implementing σ, see e.g. [14,Props.3.4 and 4.2]. In the notation of the previous paragraph let $\alpha \in \mathrm{Aut}(M)$ be ω-invariant and $\alpha(u(s)) = u(s)$, $s \in \mathbb{R}$. Then by [7,Lem.13.3] there is an isomorphism

$$\gamma: M_\omega \hat{\otimes} B(L^2(\mathbb{R})) \to N = M \times_{\sigma^\omega} \mathbb{R}$$

such that

(1) $$\tilde{\alpha} = \gamma \circ (\alpha|_{M_\omega} \otimes \iota) \circ \gamma^{-1},$$

or rather, since $\alpha = (\alpha|_{M_\omega})^{\sim}$,

(2) $$\tilde{\alpha} = (\alpha|_{M_\omega})^{\approx}.$$

Proposition 5.4. Let M be an infinite factor with separable predual. Let ω be a dominant weight and $N = M \times_{\sigma^\omega} \mathbb{R}$. Let $\alpha \in \mathrm{Aut}(M)$ Then $\tilde{\alpha}$ is inner in $\mathrm{Aut}(N)$ if and only if there are $v \in U(M)$ and an extended modular automorphism $\bar{\sigma}^\omega_c$ of M such that

$$\alpha = \mathrm{Ad}v \circ \bar{\sigma}^\omega_c.$$

Proof. We know $\tilde{\alpha}$ is inner if and only if $(\alpha \circ \mathrm{Ad}u)^{\sim}$ is inner for $u \in U(M)$. Since there is $u \in U(M)$ such that $\omega \circ \alpha = u\omega u^*$ [4, Thm. II. 1.1], we may replace α by $\alpha \circ \mathrm{Ad}u$ and assume ω is α-invariant. As in the proof of [7, Prop. 13.1] it follows from [4, p. 569] that there exists $b \in U(M_\omega)$ such that $\mathrm{Ad}b \circ \alpha(u(s)) = u(s)$, $s \in \mathbb{R}$, hence by (2)

$$\mathrm{Ad}\pi(b) \circ \tilde{\alpha} = (\mathrm{Ad}b \circ \alpha)^{\sim} = (\mathrm{Ad}b \circ \alpha|_{M_\omega})^{\approx}.$$

Thus by (1) applied to $\mathrm{Ad}b \circ \alpha$, $\tilde{\alpha}$ is inner if and only if $\alpha|_{M_\omega}$ is inner.

Assume $\tilde{\alpha}$ is inner. Then $\alpha|_{M_\omega}$ is inner, so there is $u \in U(M_\omega)$- such that $\mathrm{Ad}u \circ \alpha|_{M_\omega} = \iota$, hence by the discussion preceding the propo sition, $\mathrm{Ad}u \circ \alpha = \bar{\sigma}^\omega_c$ for a cocycle c. Thus $\alpha = \mathrm{Ad}v \circ \bar{\sigma}^\omega_c$, as asserted.

To show the converse it suffices to consider the case $\alpha = \bar{\sigma}^\omega_c$. Choose as above $b \in U(M_\omega)$ such that $\mathrm{Ad}b \circ \alpha(u(s)) = u(s)$, $s \in \mathbb{R}$. Since $\alpha|_{M_\omega} = \bar{\sigma}^\omega_c|_{M_\omega} = \iota$, $\mathrm{Ad}b \circ \alpha|_{M_\omega} = \mathrm{Ad}b|_{M_\omega}$ is inner. Thus by the first paragraph of the proof $\mathrm{Ad}\pi(b) \circ \tilde{\alpha}$, and hence $\tilde{\alpha}$, is inner.

QED

If we combine the above proposition with Theorem 5.2, we obtain the following characterization of the pointwise inner automorphisms.

Theorem 5.5. Let M be a factor of type III_λ, $0\leq\lambda<1$, with separable predual. Let ω be a dominant weight and $\alpha \in Aut(M)$. Then α is pointwise inner if and only if there are $v \in U(M)$ and an extended modular automorphism $\bar{\sigma}^{\omega}_{c}$ such that

$$\alpha = \mathrm{Ad}v \circ \bar{\sigma}^{\omega}_{c}.$$

We remark that if $0<\lambda<1$ the above theorem restricts to Corollary 3.4.

In [4,IV.2] Connes and Takesaki defined a cohomology group $H^1(F^M) = Z^1(F^M)/B^1(F^M)$, where $B^1(F^M)$ is the set of coboundaries in $Z^1(F^M)$. By [4,Cor.IV.2,5],

$$H^1(F^M) \simeq \varepsilon(\{\bar{\sigma}^{\omega}_{c} : c \in Z^1(F^M)\}).$$

where ω is any integrable weight. In particular, with ω dominant we thus have from Theorem 5.5 and Proposition 4.2,

Corollary 5.6. Let M be a factor of type III_λ, $0\leq\lambda<1$, with separable predual. Let ϕ be a faithful normal semifinite lacunary weight of infinite multiplicity on M. Then

$$H^1(\mathbb{Z},U(C_\phi)) \simeq H^1(F^M).$$

This corollary could also have been deduced from [4,Appendix] (see also [13,Thm.3.1]) by proper measure theoretic translations of the theory of the flow of weights.

Conjecture. Let M be a factor of type III_1 with separable predual. Let ϕ be a faithful normal strictly semifinite weight. Then an automorphism α of M is pointwise inner if and only if there are $u \in U(M)$ and $t \in \mathbb{R}$ such that

$$\alpha = \mathrm{Ad}u \circ \sigma^{\phi}_{t}.$$

6 The nonseparable case

We show that in the nonseparable case pointwise inner automorphisms are not as well behaved as in the separable case. Explicitly we shall exhibit outer automorphisms of II_1-factors which are pointwise inner. If τ is a trace on a II_1-factor M then $\|x\|_2 = \tau(x^*x)^{\frac{1}{2}}$ for $x \in M$.

Lemma 6.1. Let M be a factor of type II_1 with a finite normal trace τ. Let $\alpha \in \mathrm{Aut}(M)$, and let $\varepsilon > 0$ and $x \in M^+$. Then there is $u = u(x,\varepsilon) \in U(M)$ such that

$$\|\alpha(x)-uxu^*\|_2 < \varepsilon.$$

Proof. Let $\phi = \tau(x^2\cdot) \in M_*^+$. From the proof of [7,Theorem 12.4] α is approximately pointwise inner, hence there is $u = u(x,\varepsilon) \in U(M)$ such that

$$\|\phi\circ\alpha^{-1}-u\phi u^*\| < \varepsilon^2.$$

Since τ is α-invariant being the unique trace, $\tau(x^2\alpha^{-1}(y)) = \tau(\alpha(x^2)y)$. Thus we have

$$|\tau(\alpha(x^2)y)-\tau(ux^2u^*y)| < \varepsilon^2\|y\|, \qquad y \in M.$$

Hence

$$\|\alpha(x)^2-(uxu^*)^2\|_1 < \varepsilon^2.$$

By an inequality of Powers and the second author, see [10,Lem.4.1],

$$\|\alpha(x)-uxu^*\|_2 \leq \|\alpha(x)^2-(uxu^*)^2\|_1^{\frac{1}{2}} < \varepsilon,$$

QED

Let M be a factor of type II_1 with separable predual. Let ω be a free ultrafilter on $\mathbb{N}$. Let

$$I_\omega = \{(x_n)\in \ell^\infty(\mathbb{N},M): \|x_n\|_2 \underset{\omega}{\to} 0\}$$

$$M_\omega = \ell^\infty(\mathbb{N},M)/I_\omega$$

Then M_ω is known to be a II_1-factor, see e.g. [8]. Denote by π_ω the canonical homomorphism of $\ell^\infty(\mathbb{N},M)$ onto M_ω. If $x = (x_n) \in \ell^\infty(\mathbb{N},M)$ let $\bar{x} = \pi_\omega((x_n)) \in M_\omega$, and if $\alpha \in \mathrm{Aut}(M)$ let $\alpha_\omega \in \mathrm{Aut}(M_\omega)$ be defined by

$$\alpha_\omega(\bar{x}) = \pi_\omega((\alpha(x_n))).$$

<u>Theorem 6.2</u>. In the above notation if $\alpha \in \mathrm{Aut}(M)$ is outer and not approximately inner, then α_ω is pointwise inner and outer.

<u>Proof</u>. We first show α_ω is pointwise inner. Let $\bar{x} = \pi_\omega((x_n)) \in M_\omega^+$. We can assume all $x_n \in M^+$. By Lemma 6.1 there exists for each $n \in \mathbb{N}$, $u_n \in U(M)$ such that

$$\|\alpha(x_n)-u_nx_nu_n^*\|_2 < \frac{1}{n}.$$

Let $\bar{u} = \pi_\omega((u_n))$. Then $\bar{u} \in U(M_\omega)$ and

$$\bar{u}\bar{x}\bar{u}^*-\alpha_\omega(\bar{x}) = \pi_\omega((u_nx_nu_n^*-\alpha(x_n))) = 0.$$

Thus by Lemma 2.2 α_ω is pointwise inner.

We next show α_ω is outer in $\mathrm{Aut}(M_\omega)$. If not there is $\bar{u} = \pi_\omega((u_n)) \in U(M_\omega)$ such that $\alpha_\omega = \mathrm{Ad}\bar{u}$. Now if π is a homomorphism of a unital C^*-algebra A onto a von Neumann algebra N then for each $u \in U(N)$ there is $v \in U(A)$ such that $\pi(v) = u$. Indeed there is $h \in N^+$ such that $u = \exp(ih)$. Choose $k \in A^+$ such that $\pi(k) = h$. Then $\pi(\exp(ik)) = u$. We can thus assume each $u_n \in U(M)$. Let $x^1,\dots,x^m \in M^+$. Let $x_n^k = x^k$, $n \in \mathbb{N}$. Then $\alpha_\omega(\bar{x}^k) = \bar{u}\bar{x}^k\bar{u}^*$, whence

$$\lim_\omega \|u_nx^ku_n^*-\alpha(x^k)\|_2 = 0, \qquad k=1,\dots,m,$$

showing that α is approximately inner, contrary to assumption.

QED

References

1. H. Araki, Some propertice of modular conjugation operator of von Neumann algebras and a noncommutative Randon-Nikodym theorem with a chain rule, Pacific J. Math. 50 (1974), 309-354.

2. A. Connes, Une classification des facteurs de type III, Ann. Sci. École Norm. Sup. 4, Ser. 6(1973), 133-252.

3. A. Connes, Almost periodic states and factors of type III_1, J. Funct. Anal. 16(1974), 415-445.

4. A. Connes and M. Takesaki, The flow of weights on factors of type III, Tôhoku Math. J. 29(1977), 473-575.

5. S. Doplicher, D. Kastler, and E. Størmer, Invariant states and asymptotic abelianness, J. Funct. Anal. 3(1969). 419-434.

6. U. Haagerup, On the dual weights for crossed products of von Neumann algebras, II, Math. Scand. 43(1978), 119-140.

7. U. Haagerup and E. Størmer, Equivalence of normal states on von Neumann algebras and the flow of weights, Advances in Math., to appear.

8. D. McDuff, Central sequences and the hyperfinite factor, Proc. London Math. Soc. XXI(1970), 443-461.

9. S. Popa, Singular maximal abelian $*$-subalgebras in continuous von Neumann algebras, J. Funct. Anal. 50(1983), 151-166.

10. R.T. Powers and E. Størmer, Free states of the canonical anti-commutation relations, Commum. Math. Phys. 16(1970), 1-33.

11. S. Stratila, Modular theory in operator algebras, Abacus Press 1981.

12. C. Sutherland, Notes on orbit equivalence; Krieger's theorem. Lecture Notes Ser. Dept. of Math. Univ. of Oslo, No. 23(1976).

13. C. Sutherland and M. Takesaki, Actions of discrete amenable groups on injective factors of type III_λ, $\lambda \neq 1$. To appear.

14. M. Takesaki, Duality for crossed products and the structure of von Neumann algebras of type III, Acta Math. 131(1973), 249-310.

Appendix: Non-smoothness of the cohomology group $H^1(\mathbf{Z}, U(C_\phi))$

by

Colin Sutherland

It is shown that the cohomology groups $H^1(\mathbf{Z}, U(C_\phi))$ and $H^1(F^M)$ in the previous discussion are non-smooth Borel spaces. This will follow from a more general result stated below.

We let (X, μ) be a standard non-atomic measure space, and we let $\mathcal{U}$ denote the unitary group in $L^\infty(X, \mu)$. With an ergodic action of $\mathbf{Z}$ on (X, μ) we let Z^1, B^1 and H^1 be the usual spaces of 1-cocycles, coboundaries and the quotient cocycles/coboundaries with the topology of convergence in measure (as in [4,pp.18,19,24]).

<u>Theorem</u>. For any ergodic action of $\mathbf{Z}$ on (X, μ), $H^1(\mathbf{Z}, \mathcal{U})$ is not smooth.

The proof will be accomplished by showing the existence of a Borel homomorphism $\mathbf{R} \to Z^1(\mathbf{Z}, \mathcal{U})$ by $\lambda \to d^\lambda$, with $d^\lambda \in B^1(\mathbf{Z}, \mathcal{U})$ if and only if $\lambda \in \alpha\mathbf{Z} + 2\pi\mathbf{Z}$, where $\alpha \notin 2\pi\mathbf{Q}$ is given in advance. Then non-smoothness of H^1 now follows from that of $\mathbf{R}/\alpha\mathbf{Z} + 2\pi\mathbf{Z}$ (which in turn follows from [3,Thm.7.2,p.148]). We divide the proof into three parts.

(1) Observe that by [1,p.309] $H^1(\mathbf{Z}, \mathcal{U}) \simeq H^1_\mu(\mathcal{R}, \mathbf{T})$, where $\mathcal{R}$ is the equivalence relation on (X, μ) generated by the $\mathbf{Z}$-action, and this isomorhpism is derived from an isomorphism of $Z^1(\mathbf{Z}, \mathcal{U})$ with $Z^1(\mathcal{R}, \mathbf{T})$ carrying $B^1(\mathbf{Z}, \mathcal{U})$ onto $B^1(\mathcal{R}, \mathbf{T})$. (The point of doing this is that we may work with any realization of $\mathcal{R}$ that we choose.)

(2) <u>The case $\mathcal{R} = \mathcal{R}_0$, the hyperfinite II_1-relation</u>

Let $\alpha \in \mathbf{R}$, $\alpha \notin 2\pi\mathbf{Q}$, and realize $\mathcal{R}_0$ as the relation on $\mathbf{T}$ generated by the transformation $Tz = e^{i\alpha}z$. Define $d^\lambda : \mathcal{R} \to \mathbf{T}$ for

$\lambda \in \mathbf{R}$ by

$$d^\lambda(T^n z, z) = e^{in\lambda},$$

and observe $d^\lambda \in Z^1(\mathcal{R},\mathbf{T})$. (Note that in the $Z^1(\mathbf{Z},\mathcal{U})$ picture $d^\lambda(n) = e^{in\lambda}1$, where $1 \in \mathcal{U}$ is the constant function.)
Note that d^λ cobounds if and only if

$$d^\lambda(1) = c(Tz)/c(z) \quad \text{a.e.}$$

for some $c \in \mathcal{U}$, which is equivalent to saying

$$c(z)e^{i\lambda} = c(Tz) \quad \text{a.e.}$$

or to

$$c(z)e^{i\lambda} = c(e^{i\alpha}z) \quad \text{a.e.}$$

Let $c = \sum_{n\in\mathbf{Z}} c_n z^n$ be the Fourier expansion of c (with convergence in L^2). Then the above equivalence holds if and only if

$$\sum e^{i\lambda} c_n z^n = \sum e^{in\alpha} c_n z^n, \quad \text{a.e.}$$

so that

$$(*) \qquad c_n e^{i\lambda} = c_n e^{in\alpha}, \quad n \in \mathbf{Z}.$$

Thus, if $c_{n_0} \neq 0$, $e^{i\lambda} = e^{in_0\alpha}$, so that $\lambda \in \alpha Z + 2\pi Z$.

Conversely, if $\lambda \in \alpha\mathbf{Z} + 2\pi\mathbf{Z}$ then $(*)$ has a solution, so d^λ cobounds, and we have shown d^λ is a coboundary if and only if $\lambda \in \alpha\mathbf{Z} + 2\pi\mathbf{Z}$, from which it follows that $H^1(\mathcal{R},\mathbf{T})$ is not smooth. (This idea is taken from [2,p.686].)

(3) <u>The case of general</u> $\mathcal{R}$

By Krieger's theorem

$$\mathcal{R} \simeq \mathcal{R} \times \mathcal{R}_0 = \{((x,z),(x',z')) : (x,x') \in \mathcal{R}, (z,z') \in \mathcal{R}_0\},$$

so that $H^1_\mu(\mathcal{R},\mathbf{T}) \simeq H^1_\mu(\mathcal{R} \times \mathcal{R}_0, \mathbf{T})$. Define

$$\underset{\sim}{d}^\lambda((x,z),(x',z')) = d^\lambda(z,z'),$$

and note that $\lambda \to \underset{\sim}{d}^{\lambda} \in Z^1(\mathcal{R} \times \mathcal{R}_0, \mathbf{T})$ is a homomorphism. Clearly, if d^{λ} cobounds, so does $\underset{\sim}{d}^{\lambda}$.

Conversely, if

$$\underset{\sim}{d}^{\lambda}((x,z),(x',z')) = f(x,z)/f(x',z') \quad \text{a.e.}$$

then, taking $z = z'$, we get $f(x,z) = f(x',z)$ a.e. for each z, so by ergodicity, $f(x,z) = g(z)$ a.e., and so $d^{\lambda}(z,z') = g(z)/g(z')$ a.e. But this means that d^{λ} cobounds, so by case (2) $\underset{\sim}{d}^{\lambda}$ is a coboundary if and only if $\lambda \in \alpha\mathbf{Z} + 2\pi\mathbf{Z}$, and again $H^1_{\mu}(\mathcal{R},\mathbf{T})$ is not smooth.

QED

<u>Remark</u>. Essentially the same argument shows that if we have an amenable ergodic action of a locally compact group G on (X,μ), then $H^1(G,\mathcal{U})$ is not smooth. In particular, $H^1(G,\mathcal{U})$ is not smooth for any properly ergodic action of an amenable group G on (X,μ).

<u>References</u>

1. <u>J. Feldman</u> and <u>C.C. Moore</u>, Ergodic equivalence relations, cohomology and von Neumann algebras, I. Trans. Amer. Math. Soc. 234(1977), 289-324.

2. <u>S. Kawakami</u>, Irreducible representations of non-regular semidirect product groups, Math. Japonica, 26(1981), 667-693.

3. <u>G.W. Mackey</u>, Borel structures in groups and their duals, Trans. Amer. Math. Soc. 85(1957), 134-165.

4. <u>C.C. Moore</u>, Group extensions and cohomology of locally compact groups, III, Trans. Amer. Math. Soc. 221(1976), 1-33.